Constructing Fresnel reflection coefficients by ruler and compass

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A simple and intuitive geometical method to analyze Fresnel formulas is presented. It applies to transparent media and is valid for perpendicular and parallel polarizations. The approach gives a graphical characterization particularly simple of the critical and Brewster angles. It also provides an interpretation of the relation between the reflection coefficients for both basic polarizations as a symmetry in the plane.

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I. INTRODUCTION

The reflection of a plane wave at a planar interface between two homogeneous and isotropic media is a well-known phenomenon. From a general perspective, that embraces all kinds of waves, the physics of reflection is well understood: mismatched impedances generate the reflected and transmitted waves, while the application of the proper boundary conditions at the discontinuity provides the corresponding amplitude coefficients[1].

For light waves the impedance is proportional to the refractive index. Therefore, the mismatching of impedances gives Snell's law, which, in fact, is independent of the precise form of boundary conditions. On the other hand, the continuity across the boundary of the tangential components of the electric and magnetic fields yields the reflection and transmission amplitudes, which constitute the famous Fresnel formulas[2,3,4].

In view of their simplicity and elegance, it seems difficult to say anything new about Fresnel formulas. However, a quick look at the index of, e.g., this journal[5,6,7,8,9,10,11,12,13,14,15,16], immediately reveals a steady flow of papers devoted to subtle aspects of this problem, which shows that the topic is far richer than one might naively expect.

Fresnel formulas provide complete optical information about the interface. Although it is possible to fully examine their physical implications by a purely algebraic analysis, the discussion is usually carried out by using plots of the coefficients in terms of the angle of incidence. This is mainly due to the belief that graphical results convey information more readily than algebraic formulas. In this spirit, it has also been proposed the use of geometrical methods[17,18,19] to analyze the problem. In our opinion, these geometrical approaches are still worth exploring in order to take full advantage of them. Thus, in this paper we propose a new and extremely simple method that allows one to construct Fresnel formulas by ruler and compass. Geometrical construction by ruler and compass is a fascinating problem since ancient times, and offers the additional advantage for the students of easily visualizing the steps of any construction and how it varies for different inputs.

We emphasize that these methods do not offer any inherent advantage in terms of computational efficiency. Apart from their beauty, their benefit lies in the possibility of gaining insights into the qualitative behavior of the Fresnel coefficients, which is important in developing a physical feeling of these relevant equations.

II. FRESNEL FORMULAS

Let two homogeneous isotropic semi-infinite media, described by complex refractive indices N_0 and N_1, be separated by a plane boundary. We assume an incident monochromatic, linearly polarized plane wave from medium 0, which makes an angle θ_0 with the normal to the interface and has amplitude E^i. The electric field is either in the plane of incidence (denoted by subscript $\parallel$) or perpendicular to the plane of incidence (subscript $\perp$). This wave splits into a reflected wave E^r in medium 0, and a transmitted wave E^t in medium 1 that makes an angle θ_1 with the normal. The angles of incidence θ_0 and refraction θ_1 are related by Snell's law:

$$N_0 \sin\theta_0 = N_1 \sin\theta_1 . \tag{1}$$

If media 0 and 1 are transparent (so that N_0 and N_1 are real numbers) and no total reflection occurs, the angles θ_0 and θ_1 are also real and the above picture of how a plane wave is reflected and refracted at the interface is simple. However, when either one or both media is absorbing, the angles θ_0 and θ_1 become, in general, complex and the discussion continues to hold only formally, but the physical picture of the fields becomes complicated[20].

The wave vectors of all waves lie in the plane of incidence and when the incident fields are $\perp$- or $\parallel$-polarized, all plane waves excited by the incident ones have the same polarization[21]. An arbitrarily polarized incident wave can be resolved into its $\perp$ and $\parallel$ components, and each of them can be treated separately.

By demanding that the tangential components of $\mathbf{E}$ and $\mathbf{H}$ should be continuous across the boundary[2], and assuming nonmagnetic media, the reflection and transmission amplitudes are given by[22]

$$r_\perp = \frac{E_\perp^r}{E_\perp^i} = \frac{N_0\cos\theta_0 - N_1\cos\theta_1}{N_0\cos\theta_0 + N_1\cos\theta_1}, \tag{2a}$$

$$r_\parallel = \frac{E_\parallel^r}{E_\parallel^i} = \frac{N_1\cos\theta_0 - N_0\cos\theta_1}{N_1\cos\theta_0 + N_0\cos\theta_1}, \tag{2b}$$

$$t_\perp = \frac{E^t_\perp}{E^i_\perp} = \frac{2N_0 \cos\theta_0}{N_0 \cos\theta_0 + N_1 \cos\theta_1}, \tag{2c}$$

$$t_\parallel = \frac{E^t_\parallel}{E^i_\parallel} = \frac{2N_0 \cos\theta_0}{N_1 \cos\theta_0 + N_0 \cos\theta_1}, \tag{2d}$$

which are the Fresnel formulas. The physical contents of these coefficients are discussed in any optics textbook.

III. GRAPHICAL CONSTRUCTION FOR THE REFLECTION COEFFICIENTS

A. General considerations

The geometrical method we outline here deals only with the case when media 0 and 1 are transparent. The refractive indices are then real numbers that we denote by n_0 and n_1. Before going into details we wish to show how the algebraic properties of the Fresnel reflection coefficients can be intimately linked to some elementary geometrical properties of an isosceles trapezoid.

The key point for our purposes is to note that the reflection coefficients can be written as

$$r = \frac{a-b}{a+b}, \tag{3}$$

where a and b are positive real numbers whose explicit form depend on the polarization. To visualize the algebraic properties of this quotient[23] we propose to use the geometrical construction depicted in Fig. 1. Let us assume first (Fig. 1.a) that $a > b$: then $r > 0$ and this coefficient can be inferred from an isosceles trapezoid with minor basis $a-b$ and major basis $a+b$, since the quotient of such bases is precisely r. When $a = b$ (Fig. 1.b) the trapezoid degenerates in an isosceles triangle and $r = 0$. Finally, when $a < b$, the numerator in r is negative. Since a segment cannot have a negative length, we represent this case by plotting the bow tie of Fig. 1.c and we have then that $r < 0$.

In the next section we shall illustrate how this simple construction allows for a complete determination of the reflection coefficients. Before we do that and to relate this interpretation with other previous results, we wish to note that any equation of the form (3) can always be expressed as

$$r = \frac{a-b}{a+b} = \frac{\xi - 1/\xi}{\xi + 1/\xi} = \tanh\zeta, \tag{4}$$

where

$$\xi = \sqrt{\frac{a}{b}} = \exp(\zeta). \tag{5}$$

In spite of its simplicity, we think that this is a remarkable formula[24,25,26]. It states that any reflection coefficient can be always expressed as a hyperbolic tangent, just as in special relativity the velocities are expressed in terms of the rapidity[27,28,29].

B. Perpendicular polarization

The previous general reasoning suggests that Fresnel formulas can be seen from a purely geometrical viewpoint. To go one step further, and to quantify these ideas, we plot two concentric circumferences of radii n_0 and n_1 centered at the origin O, as shown in Fig. 2 (we need to take care only of the first quadrant). We assume $n_0 < n_1$, which does not suppose any serious restriction.

For the case of polarization $\perp$ that we are considering, we draw in the circumference n_0 the radius that forms an angle θ_0 with the horizontal. We denote by S_0 the point where this radius intersects the circumference. Next, we draw a horizontal line from S_0 that intersects the circumference n_1 at the point S_1. The radius OS_1 determines then the refraction angle θ_1. A quick look at this figure shows that the projections on the vertical axis of the radii OS_0 and OS_1 are identical, in agreement with Snell's law.

If the light is incident from medium 1 at angle θ_1 we would obtain first S_1 and, in a completely analogous way, the corresponding pair S_0 and θ_0.

When the point S_0 runs the quadrant n_0 (i.e., θ_0 varies from 0 to $\pi/2$) the point S_1 runs the quadrant n_1 in such a way that θ_1 varies from 0 to the critical angle θ_c (obviously, total internal reflection occurs only when the light is incident from the denser medium). This is a clear way of picturing the critical angle.

On the other hand, the projections on the horizontal axis of the radii OS_0 and OS_1 measure $a = n_0 \cos\theta_0$ and $b = n_1 \cos\theta_1$, respectively. The values of $a-b$ and $a+b$ have been marked in Fig. 3 as bold segments. Since $a < b$ for every θ_0, $r_\perp$ is always negative and the trapezoid is a bow tie, according to our previous discussion. For $\theta_0 = 0$ the bow tie degenerates into two horizontal segments of lengths $n_1 - n_0$ and $n_1 + n_0$, while for $\theta_0 = \pi/2$ it reduces to two identical triangles.

As the angle of incidence increases, the minor basis grows while the major basis decreases. Therefore, the absolute value of $r_\perp$ grows monotonically with the angle of incidence, up to its maximum value of 1 at grazing incidence.

Finally, the numerical value of $r_\perp$ can be determined by a ruler. To this end, it suffices with constructing the right-angled triangle having as legs the bases of the previous bow tie. It is obvious that $r_\perp$ coincides with the height of a similar triangle of unity basis, as shown in Fig. 3.

C. Parallel polarization

The richer phenomenology of the $\parallel$ polarization is clearly highlighted also by this graphical construction. The method is essentially the same as that for $\perp$ polarization. Once we are given the incidence angle θ_0, we construct the refraction angle θ_1 much in the same way.

It is clear from Eqs. (2a) and (2b) that $r_\parallel$ can be obtained from $r_\perp$ by exchanging the refractive indices. In consequence, this suggests, as shown in Fig. 4, that now the radius OS_0 must be extended until it intersects the circumference n_1 at the point P_0. Similarly, we obtain the point P_1 as the intersection of the radius OS_1 with the circumference n_0. Obviously, the horizontal projections of OP_0 and OP_1 are $a = n_1 \cos\theta_0$ and $b = n_0 \cos\theta_1$, respectively and, as before, $a - b$ and $a + b$ have been marked in Fig. 4 as bold segments. These segments are the bases of an isosceles trapezoid that appears shaded in the figure. The nonparallel sides have a length n_0 and form an angle θ_1 with the major basis.

For $\theta_0 = 0$ the trapezoid degenerates into two horizontal segments of lengths $n_1 - n_0$ and $n_1 + n_0$, as it happens for $\perp$ polarization. This reproduces the well-known fact that at normal incidence there is no physical difference between both basic polarizations, except for a sign.

As θ_0 increases, the minor basis P_1P_0' decreases, reaching the value 0 (so $r_\parallel = 0$ also), which defines the Brewster angle. In this particular angle ($\tan\theta_B = n_1/n_0$) the points P_0 and P_1 are on the same vertical and, therefore, the trapezoid becomes a triangle. We think that this provides a remarkable way of visualizing this important angle (see Fig. 5). When the angle of incidence is further increased, $r_\parallel$ becomes negative. This can be seen in the fact that the trapezoid becomes a bow tie. Finally, at grazing incidence this bow tie is made from two identical isosceles triangles and then $r_\parallel = -1$, as for the $\perp$ polarization. The numerical value of $r_\parallel$ can be determined by a ruler by the same procedure as before.

Before finishing, we wish to emphasize that, concerning the behavior of the corresponding transmission coefficients, this geometrical construction still applies, but its interpretation is more involved, because they do not behave like a hyperbolic function[28].

D. Relation between basic polarizations

It is the purpose of this section to show how our approach interprets the quotient[3]

$$\frac{r_\perp}{r_\parallel} = -\frac{\cos(\theta_0 - \theta_1)}{\cos(\theta_0 + \theta_1)} \tag{6}$$

as a simple symmetry in the plane. Azzam[30,31] has noticed that this relation can be recast so as to show a universal character (i.e, it is independent of the two media that define the interface), and it has proven to be of practical interest.

Let us denote by $\{AB\}$ the horizontal projection of the segment AB. From our previous discussion, we can recast the reflection coefficients as

$$r_\perp = \frac{\{OS_0 - OS_1\}}{\{OS_0 + OS_1\}} = \frac{S_1S_0}{\{OS\}}, \tag{7a}$$

$$r_\parallel = \frac{\{OP_0 - OP_1\}}{\{OP_0 + OP_1\}} = \frac{\{P_1P_0\}}{\{OP\}}, \tag{7b}$$

where we have written $OS_0 + OS_1 = OS$ and $OP_0 + OP_1 = OP$. It is easy to check that OS and OP have the same length.

Next, we note that the four points considered until now, namely, S_0, S_1 and P_0, P_1, are the vertex of the sector of annulus shaded in Fig. 6. The triangles OP_0P_1 and OS_0S_1 are identical and can be obtained by a reflection through the bisecting line OB of the sector. The two diagonals S_0S_1 and P_0P_1 have the same length and form between them an angle $\theta_0 + \theta_1$. On the other hand, the bisecting line OB forms with the horizontal axis an angle $(\theta_0 + \theta_1)/2$ and it is a symmetry axis with respect to the pairs of points S_0 and P_1, P_0 and S_1, and P and S. That is, OP and OS form the same angle α with OB, where α can be computed to be

$$\begin{aligned} \tan\alpha &= \frac{(n_1 - n_0)\sin[(\theta_0 - \theta_1)/2]}{(n_1 + n_0)\cos[(\theta_0 - \theta_1)/2]} \\ &= \frac{n_1 - n_0}{n_1 + n_0}\tan[(\theta_0 - \theta_1)/2]. \end{aligned} \tag{8}$$

We thus conclude that the numerators in (7a) and (7b) are related by

$$\frac{S_1S_0}{\{P_1P_0\}} = \frac{1}{\cos(\theta_0 + \theta_1)}, \tag{9}$$

while for the denominators it holds

$$\frac{\{OP\}}{\{OS\}} = \frac{\cos[(\theta_0 + \theta_1)/2 + \alpha]}{\cos[(\theta_0 + \theta_1)/2 - \alpha]} = \cos(\theta_0 - \theta_1). \tag{10}$$

This concludes the geometrical proof of Eq. (6) and shows that the coefficients $r_\perp$ and $r_\parallel$ can be interpreted as quotients of horizontal projections of segments that are symmetric with respect to the bisecting line. As far as we know, this simple result has been never noticed in the literature.

IV. CONCLUDING REMARKS

The method given in this paper provides a geometrical construction for getting the values of the Fresnel reflection coefficients by ruler and compass. These graphical methods appeal more readily to students than an analytical treatment.

The construction allows one to show at a simple glance the variation of these coefficients with the angle of incidence and the peculiarities of the parallel polarization when crossing the Brewster angle.

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FIG. 1: Geometrical interpretation of the Fresnel reflection coefficients as the quotient of two segments: a) for $a > b$ we have an isosceles trapezoid and $r > 0$; b) for $a = b$ the trapezoid becomes a triangle and $r = 0$; c) for $a < b$ then $r < 0$ and, by continuity, we represent now the trapezoid as a bow tie.

FIG. 2: Illustrating how to obtain the angle of refraction θ_1 from the angle of incidence θ_0. The appearance of the critical angle θ_c is evident.

FIG. 3: Sketch of the method for getting the reflection coefficient $r_\perp$ by ruler and compass.

FIG. 4: Construction of the trapezoid associated to the reflection coefficient $r_\parallel$ below the Brewster angle.

FIG. 5: Same as in Fig. 4, but for the Brewster angle.

FIG. 6: Showing the relation between the reflection coefficients for both basic polarizations as a symmetry in the plane.